A Stochastic Theory of Fatigue Crack Propagation

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A new mathematical theory is proposed to analyze the propagation of fatigue crack based on the concepts of fracture mechanics and random processes. The time-dependent crack size is approximated by a Markov process. Analytical expressions are obtained for the probability distribution of crack size at any given time and the probability distribution of the random time at which a given crack size is reached, conditional on the knowledge of the initial crack size. Examples are given to illustrate the application of the theory, and the results are compared with available experimental data.

Introduction

FROM a fracture mechanics point of view, the fatigue damage of a structural component subjected to dynamic loads can be measured by the size of the dominant crack, and failure occurs when this crack reaches a critical magnitude. A number of mathematical functions have been proposed for the fatigue crack propagation rate. These functions have the general form of (see, for example, Refs. 1 and 2):

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = Q(K, \Delta K, s, \underline{a}, R) \tag{1}$$

where $\underline{a}(t)$ is the crack size at time t, Q a non-negative function, K the stress intensity factor, ΔK the stress intensity range, s the stress amplitude, and R the stress ratio. However, even in a well controlled laboratory environment, results obtained from crack growth experiments under either a constant-amplitude cyclic loading or a given spectrum loading usually exhibit considerable statistical variability. Therefore, statistical analyses are quite appropriate for such problems (e.g., Refs. 3-21).

If we restrict our attention to a laboratory setting so that the loading time variation is deterministic, then a mathematical model of the form of Eq. (1) can be "randomized" as follows:

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = Q(K, \Delta K, s, \underline{a}, R)X(t) \tag{2}$$

where the added factor $X(t)$ is a non-negative random process. It is of interest to note that Virkler et al.[13,14] have undertaken simulation studies of crack propagation which amounted to assuming $X(t)$ in Eq. (2) to be totally independent at any two different times. At the other extreme, Yang et al.[19-21] has used a random variable instead of a random process $X(t)$ in Eq. (2), which is equivalent to assuming a totally correlated $X(t)$ at all times. It was pointed out in Ref. 20 that a totally independent $X(t)$ would lead to the smallest statistical dispersion and a totally correlated $X(t)$ to the greatest statistical dispersion for crack propagation. Actual experimental data suggest that a more realistic modeling of fatigue crack growth should lie somewhere between the two extremes.

The physical meaning of the added random factor $X(t)$ is clear. It represents the combined effect of unknown contributions toward changing the crack propagation rate with time. Within the general class of Eq. (2) the simplest mathematical model is a totally independent $X(t)$ at any two different times. In this case the crack size $\underline{a}(t)$ becomes a continuously parametered Markov process. However, this is unconservative because the extent of statistical scatter would be underestimated.

Let us suppose that Eq. (2) is a good representation of the crack propagation mechanism, and the unknown $\underline{a}(t)$ is now treated as a random process. The difference between this random $\underline{a}(t)$ and the one obtained from the deterministic equation, Eq. (1), to be called the random deviation in $\underline{a}(t)$, is smoother than $X(t)$ which is the cause for the deviation. Statistically, $\underline{a}(t)$ is said to have a longer correlation time than $X(t)$. If $\underline{a}(t)$ is observed at time intervals greater than the correlation time of $X(t)$, the observed behavior will be approximately Markovian. The above reasoning suggests that a mathematical procedure, called stochastic averaging,[23] may be applicable in the present case. The implication of the stochastic averaging procedure is to lump the effect of past correlation of the "excitation" $X(t)$ and place it at the present when obtaining a governing equation for a Markov process which is an approximation for the physical $\underline{a}(t)$ process. This idea has been used in a previous paper[24] to investigate the random duration of time required to reach any given crack size. A recursive relationship has been obtained for the statistical moments of this random time, and a specific example given for the case where crack propagation obeys a power law. The present paper will follow the same general approach but attention will be focused on the probability distribution of the random crack size at any given time instant, as well as the probability distribution of the random time to reach any given crack size referred to above.

It is interesting to note that modeling the crack size as a Markov process but discretely valued and discretely parametered, namely, a Markov chain, has been proposed by Bogdanoff[17] and Kozin and Bogdanoff.[18] In these references, the discretized time parameter is the number of duty cycles, and the increments of the crack size in two different duty cycles are assumed to be independent. There is a conceptual similarity between the Bogdanoff-Kozin approach and the

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approach presented herein; that is, the crack size process is treated as being approximately Markovian if it is observed at far enough time intervals. However, our model is fracture mechanics-based whereas the Bogdanoff-Kozin model is not. Furthermore, the application of the Bogdanoff-Kozin model requires the estimation of a large Markov transition matrix which appears to be more difficult and requires a much larger data base for crack size vs cycles. When experimental results are not plentiful, estimating just a few fracture mechanics parameters and the statistical properties, as in the present approach, should be much simpler. Another advantage in this approach is the relatively small amount of computation that is required in its application. The extension of the theory to include the more general case where the stress time variation is also random will be discussed in a future paper.

Model for $X(t)$

We shall model $X(t)$ as a random pulse train (see, e.g., Ref. 25);

$$X(t)=\sum_{k=1}^{N(t)} Z_k w(t,\tau_k) \tag{3}$$

where $N(t)$ is a homogeneous Poisson counting process, denoting the total number of pulses that arrive within the time interval $(-\infty,t)$, τ_k the arrival time of the kth pulse, Z_k the random amplitude of the kth pulse, and

$$\begin{aligned} w(t,\tau)=w(t-\tau)&=1, \qquad 0<t-\tau\leq\Delta \\ &=0, \qquad \text{otherwise} \end{aligned} \tag{4}$$

We further assume that Z_k for different k are independent, identically distributed random variables, with a common probability distribution as Z.

The statistical properties of $X(t)$ can be described by its cumulant (or semi-invariant) functions. The mth cumulant function is given by[25]:

$$\begin{aligned} &\kappa_m[X(t_1),\ldots,X(t_m)] \\ &=E[Z^m]\lambda\int_{-\infty}^{\min(t_1,\ldots,t_m)} w(t_1-\tau)\ldots w(t_m-\tau)\,d\tau \end{aligned} \tag{5}$$

in which $E[\ ]$ denotes an ensemble average, λ the average arrival rate of the Poisson process, and min ( ) indicates the smallest of the parenthesized quantities. In particular, the first cumulant is the mean function, and the second cumulant is the covariance function. These are found to be:

$$\begin{aligned} \mu=E[X(t)]&=E[Z]\lambda\int_{-\infty}^{t} w(t-\tau)\,d\tau \\ &=E[Z]\lambda\int_0^\infty w(u)\,du=E[Z]\lambda\Delta \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathrm{Cov}[X(t_1),X(t_2)]&=E[Z^2]\lambda\int_{-\infty}^{t_1} w(t_1-\tau)w(t_2-\tau)\,d\tau \\ &=E[Z^2]\lambda\int_0^\infty w(u)w(t_2-t_1+u)\,du \\ &=2\beta(1-|t_2-t_1|/\Delta), \qquad |t_2-t_1|<\Delta \\ &=0, \qquad |t_2-t_1|\geq\Delta \end{aligned} \tag{7}$$

in which $\beta=\frac{1}{2}E[Z^2]\lambda\Delta$.

Approximation of $\underline{a}(t)$ by a Markov Random Process

We now rewrite Eq. (2) as follows:

$$\frac{d\underline{a}}{dt}=Q(\underline{a})[\mu+Y(t)] \tag{8}$$

where the dependence of Q on K, ΔK, s, and R has been suppressed for simplicity. Clearly, $Y(t)$ is a random process with zero mean and the correlation function of $Y(t)$ is the same as the covariance function of $X(t)$; namely,

$$\begin{aligned} R_{YY}(\tau)=E[Y(t)Y(t+\tau)]&=2\beta(1-|\tau|/\Delta), \qquad |\tau|\leq\Delta \\ &=0, \qquad \text{otherwise} \end{aligned} \tag{9}$$

A sketch of this correlation function is shown in Fig. 1. If the correlation time of $Y(t)$ is short compared with the characteristic time of $\underline{a}(t)$, then $\underline{a}(t)$ is close to a diffusive Markov process[23] which is governed by an Itô's stochastic differential equation (see, e.g., Ref. 26)

$$d\underline{a}=m(\underline{a},t)\,dt+\sigma(\underline{a},t)\,dB(t) \tag{10}$$

where m is called the drift coefficient, σ the diffusion coefficient, and $B(t)$ a unit Brownian motion process (also called the Wiener's process), which has the property that $dB(t_1)$ and $dB(t_2)$ are independent for $t_1\neq t_2$.

The correlation time of $Y(t)$ may be defined as follows:

$$\tau_{\mathrm{cor}}=\int_0^\infty \tau|R_{YY}(\tau)|\,d\tau\Big/\int_0^\infty |R_{YY}(\tau)|\,d\tau \tag{11}$$

Substitution of Eq. (9) into Eq. (11) results in $\tau_{\mathrm{cor}}=\Delta/3$.

Strictly speaking, the $\underline{a}(t)$ process in Eq. (10) is an approximation of that in Eq. (8), and they could be represented more clearly by two different symbols; however, the same symbol will be used in this paper for both processes as long as no confusion will result.

When the Markov approximation is justified, Stratonovich's stochastic averaging method provides the required formulas to compute the drift and diffusion coefficients from the original physical equation. In the case of Eq. (8)

$$\begin{aligned} m&=Q\mu+\int_{-\tau_0}^{0} Q\frac{\partial Q}{\partial\underline{a}}E[Y(t)Y(t+\tau)]\,dt \\ &=Q\mu+Q\frac{\partial Q}{\partial\underline{a}}\beta\Delta=Q\left(\mu+\frac{\partial Q}{\partial\underline{a}}\beta\Delta\right) \end{aligned} \tag{12}$$

$$\sigma^2=2\int_{-\tau_0}^{0} Q^2E[Y(t)Y(t+\tau)]\,d\tau=2Q^2\beta\Delta \tag{13}$$

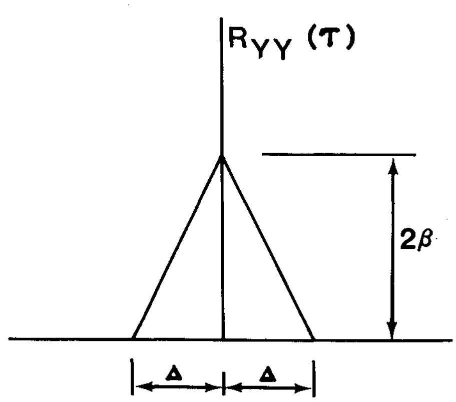

Fig. 1 Autocorrelation function of random process $Y(t)$.

where $\tau_0 > \Delta$. These equations imply that Q and $\partial Q/\partial \underline{a}$ vary slowly within the integration interval to justify their being taken outside the integrals. Thus, the integrals account basically for the contribution toward the drift and diffusion due to the correlation between the past and present "excitations." This contribution is lumped at the present, when m and σ are used in Eq. (10). The replacement of Eq. (8) by Eq. (10) amounts to substituting $Y(t)$ by a white noise. Theoretically, the substitution introduces an error associated with the probability for $d\underline{a}$ to become negative. This error is negligible as long as the tendency for drift (mean crack growth rate) dominates the tendency for diffusion (variation around mean crack growth rate), which is usually the case as seen in the examples presented herein. Stratonovich's formulas are applicable to other types of correlation functions for $Y(t)$ as long as Q and $dQ/d\underline{a}$ vary slowly within an interval of τ where such a correlation is not negligibly small. In this case, the lower limit of integration can even be extended to $-\infty$. Stratonovich's method originally was proposed on a physical ground, but later proved rigorously by Khasminski[27] and given a rigorous mathematical interpretation.

The transition probability density $q_a(a,t|a_0,t_0)$ of the Markov process $\underline{a}(t)$ is a conditional probability density which describes the distribution of $a(t)$ under the condition that the crack size is $\underline{a}(t_0)=a_0$ at an earlier time t_0. It is governed by the following Fokker-Planck equation (see, for example, Ref. 25):

$$\frac{\partial q_a}{\partial t} + \frac{\partial}{\partial a}\left[Q\left(\mu+\beta\Delta\frac{\partial Q}{\partial a}\right)q_{\underline{a}}\right] - \frac{\partial^2}{\partial a^2}(Q^2\beta\Delta q_{\underline{a}}) = 0 \tag{14}$$

or by the adjoint of Eq. (14):

$$\frac{\partial q_a}{\partial t_0} + \left[Q\left(\mu+\beta\Delta\frac{\partial Q}{\partial a_0}\right)\right]\frac{\partial q_a}{\partial a_0} + Q^2\beta\Delta\frac{\partial^2}{\partial a_0^2}q_{\underline{a}} = 0 \tag{15}$$

subject to the condition,

$$q_{\underline{a}}(a,t_0|a_0,t_0) = \delta(a-a_0) \tag{16}$$

In Eq. (14) Q is treated as a function of "a," whereas in Eq. (15) it is treated as a function of a_0. These equations are also known as Kolmogorov's forward and backward equations, respectively.

Probability Distribution of Crack Size at a Given Time

We wish to solve Eq. (14) with the least restriction on the form of the Q function, except that it must be non-negative. We shall investigate the possibility of simplifying this equation by changing the independent variable a to $z=z(a)$, and the dependent variable $q_a(a,t|a_0,t_0)$ to $q(z,t|z_0,t_0)$. For $q(z,t|z_0,t_0)$ to be a valid probability density,

$$q_{\underline{a}}(a,t|a_0,t_0) = z'q(z,t|z_0,t_0) \tag{17}$$

where $z'=\partial z/\partial a$. We shall choose the functional form $z=z(a)$ such that z' is positive. By a straightforward substitution, Eq. (14) is changed to

$$z'\frac{\partial q}{\partial t} + \left\{\mu\left(z'\frac{\partial Q}{\partial a}+z''Q\right) - \beta\Delta\frac{\partial}{\partial a}\left[Q\left(z'\frac{\partial Q}{\partial a}+z''Q\right)\right]\right\}q$$
$$+\left\{\mu Q(z')^2 - 3\beta\Delta Qz'\left(z'\frac{\partial Q}{\partial a}+z''Q\right)\right\}\frac{\partial q}{\partial z}$$
$$-\beta\Delta Q^2(z')^3\frac{\partial^2 q}{\partial z^2} = 0 \tag{18}$$

Thus, considerable simplification can be achieved if we select $z(a)$ such that

$$z'\frac{\partial Q}{\partial a} + z''Q = 0 \tag{19}$$

Equation (19) is satisfied by

$$z'Q = \text{const} \tag{20}$$

Letting this constant be equal to one,

$$z = \int_{a_0}^{a}\frac{dv}{Q(v)} \tag{21}$$

Substituting Eq. (21) into Eq. (18), we obtain

$$\frac{\partial q}{\partial t} + \mu\frac{\partial q}{\partial z} - \beta\Delta\frac{\partial^2 q}{\partial z^2} = 0 \tag{22}$$

subject to the initial condition

$$q(z,t_0|0,t_0) = \delta(z) \tag{23}$$

It is interesting to note that if we had introduced a random process at the onset,

$$Z(t) = \int_{a_0}^{\underline{a}(t)}\frac{dv}{Q(v)} \tag{24}$$

and transformed Eq. (8) into

$$dZ(t) = [\mu + Y(t)]dt \tag{25}$$

then we would have arrived at Eq. (22) much more simply, since an application of stochastic averaging to Eq. (25) would show readily that μ and $\sqrt{2\beta\Delta}$ were the drift and diffusion coefficients for the Markov approximation of the $Z(t)$ process, and that the associated Fokker-Planck equation was Eq. (22). However, the backward equation corresponding to Eq. (22), namely,

$$\frac{\partial q}{\partial t_0} + \mu\frac{\partial q}{\partial z_0} + \beta\Delta\frac{\partial^2 q}{\partial z_0^2} = 0 \tag{26}$$

does not have a physical meaning, and, in fact, is not transformable from the backward equation for the $\underline{a}(t)$ process, Eq. (15). In view of this subtle point, and the need for Eq. (15) for future developments, the lengthier derivation of Eq. (22) is retained herein.

The solution of Eq. (22) that satisfies the initial condition (23) is given by

$$q(z,t|0,t_0) = \frac{1}{\sqrt{2\pi}\sqrt{2\beta\Delta(t-t_0)}}\exp\left\{-\frac{[z-\mu(t-t_0)]^2}{4\beta\Delta(t-t_0)}\right\} \tag{27}$$

Correspondingly, the solution of Eq. (14) is

$$q_{\underline{a}}(a,t|a_0,t_0) = \frac{1}{\sqrt{2\pi}\sqrt{2\beta\Delta(t-t_0)}\,Q}$$
$$\times\exp\left\{-\left[\int_{a_0}^{a}\frac{dv}{Q(v)} - \mu(t-t_0)\right]^2\Big/4\beta\Delta(t-t_0)\right\} \tag{28}$$

It should be noted that Eq. (28) describes the probability structure of a diffusive Markov process which is an approximation to the physical crack size process $\underline{a}(t)$. In fact, Eq. (28) must also admit those a values which are smaller than a_0; otherwise, the total probability would not be equal to unity. The error which arises from restricting $a \geq a_0$ is greatest when t is near t_0, and decreases as $t-t_0$ increases. Since we are interested only in reasonably long fatigue lives, the error should be negligible. This error, however, can be compensated by renormalizing the total probability at any instant of time, namely, by changing Eq. (28) to

$$q_{\underline{a}}(a,t|a_0,t_0) = \frac{D(t-t_0)}{\sqrt{2\pi}\sqrt{2\beta\Delta(t-t_0)}\,Q} \times \exp\left\{-\left[\int_{a_0}^{a}\frac{\mathrm{d}v}{Q(v)} - \mu(t-t_0)\right]^2 \Big/ 4\beta\Delta(t-t_0)\right\}; \quad a \geq a_0 \tag{29}$$

where

$$D(t-t_0) = \frac{2}{1+\mathrm{erf}[(\mu/2\sqrt{\beta\Delta})\sqrt{t-t_0}]} \tag{30}$$

and erf(·) = the error function. When drift dominates the diffusion,

$$\mu(t-t_0) \gg \sqrt{2\beta\Delta(t-t_0)} \tag{31}$$

and $D(t-t_0) \approx 1$, in agreement with our previous remark.

We propose that Eq. (29) be used to estimate the probability distribution of the crack size at time t. Although Eq. (29) does not satisfy Eq. (14) exactly, it will be shown in our illustrative examples that this approximation compares quite well with experimental data.

The probability distribution function of the crack size $\underline{a}(t)$ at time t is given by

$$F_{\underline{a}(t)}(a,t|a_0,t_0) = P[\underline{a}(t) \leq a|a_0,t_0] = \int_{a_0}^{a} q_{\underline{a}}(x,t|a_0,t_0)\,\mathrm{d}x \tag{32}$$

and the probability that the crack size exceeds a given value a is the complement of $F_{\underline{a}(t)}(a,t|a_0,t_0)$, i.e.,

$$F^*_{\underline{a}(t)}(a,t|a_0,t_0) = P[\underline{a}(t) > a|a_0,t_0] = 1 - F_{\underline{a}(t)}(a,t|a_0,t_0) = \int_{a}^{\infty} q_{\underline{a}}(x,t|a_0,t_0)\,\mathrm{d}x \tag{33}$$

The graph of $F^*_{\underline{a}(t)}(a,t|a_0,t_0)$ is often referred to as the crack exceedance curve.

Let $T(a_l)$ be the random time when the crack size $\underline{a}(t)$ reaches a specific value a_l. Since the event $\{T(a_l) \leq \tau\}$ is the same as the event $\{\underline{a}(t_0+\tau) > a_l\}$, the probability distribution function $F_{T(a_l)}(\tau)$ of $T(a_l)$ is given by

$$F_{T(a_l)}(\tau) = \int_{a_l}^{\infty} q_{\underline{a}}(x,t_0+\tau|a_0,t_0)\,\mathrm{d}x = F^*_{\underline{a}(t_0+\tau)}(a_l,t_0+\tau|a_0,t_0) \tag{34}$$

which is the same as the probability of crack exceedance but treated as a function of τ.

Correlation with Experimental Test Results

Three examples are given to illustrate the application of the present theory: two are chosen in the small crack size range and one in the large crack size range. It is known that these two ranges are quite different and the statistical variability in the small crack size range is much greater than that in the large crack size range.

Crack Propagation in Fastener Holes

Fractographic data of 7475-T7351 aluminum fastener hole specimens subjected to a bomber load spectrum are available in Ref. 22. Figure 2 shows the crack propagation time histories for two data sets, referred to as the WPB and XWPB data sets, respectively. The letters W, P, and B indicate that the specimens were drilled with a Winslow Spacematric machine (W), using a proper drilling technique (P), and subjected to a given B-1 bomber load spectrum (B). The additional symbol X associated with the XWPB data set has the meaning that the fasteners are configured to transfer 15% load, whereas the WPB fasteners transfer no load. The fractographic data have been censored to include only those corner cracks propagating from 0.004 to 0.04 in. for the WPB data set and from 0.004 to 0.07 in. for the XWPB data set. This censoring procedure is necessary in order to normalize the data to zero life at 0.004 in., thus obtaining homogeneous data sets as shown in Fig. 2. The resulting WPB and XWPB data sets consist of 16 and 22 specimens, respectively.

Within the small crack size range, the power law for the crack growth rate is shown to be valid[8-12]; namely, $Q = Q^*\underline{a}^b$ or

$$\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = Q^*\underline{a}^b X(t) \tag{35}$$

where Q^* and b are constants.

Taking logarithms on both sides of Eq. (35), one obtains

$$\log\frac{\mathrm{d}\underline{a}}{\mathrm{d}t} = b\log\underline{a} + \log Q^* + \log X(t) \tag{36}$$

Test results of log $[\mathrm{d}\underline{a}(t)/\mathrm{d}t]$ vs log $\underline{a}(t)$ are plotted in Fig. 3 as dots for the two data sets. Since $X(t)$ is assumed to be a stationary random process, the mean value and standard deviation of $\log X(t)$ are constants.

To estimate the fracture mechanics parameters b and Q^* from the test results of the crack growth rate in Fig. 3, a linear

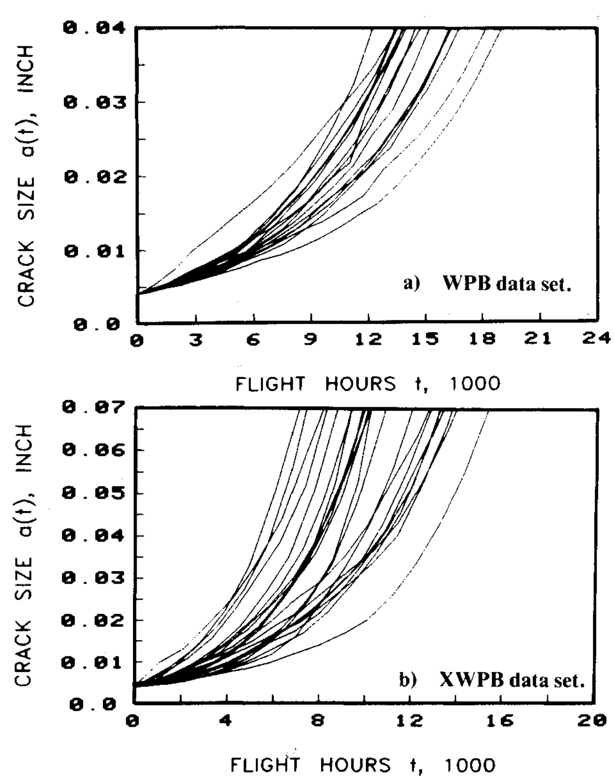

Fig. 2 Actual crack propagation time histories for fastener hole specimens.

regression analysis was carried out for $\log[\mathrm{d}\underline{a}/\mathrm{d}t]$ on $\log \underline{a}$. This implies that the distribution of $\log X(t)$ is normal with zero mean, and the linear regression also results in the standard deviation, $\sigma_{\log X(t)}$. The results for the WPB data set are

$$b=0.9297, \quad Q^*=1.1051\times 10^{-4}, \quad \sigma_{\log X(t)}=0.087635 \tag{37}$$

where the units of a and t are inches and flight hours, respectively.

The mean value, μ_X, and the standard deviation, σ_X, of $X(t)$ are obtained from $\sigma_{\log X(t)}$ using the normal-to-lognormal conversion formulas,

$$\mu_X=\exp\{\tfrac{1}{2}[\sigma_{\log X(t)}\ln 10]^2\}=\mu \tag{38}$$

$$\sigma_X^2=\mu_X^2\{\exp[(\sigma_{\log X(t)}\ln 10)^2]-1\}=2\beta \tag{39}$$

Application of Eqs. (38) and (39) yields

$$\mu=1.0206, \qquad \beta=0.021643 \tag{40}$$

For a special case in which the crack propagation rate follows a power law as shown in Eq. (35), i.e., $Q=Q^*\underline{a}^b$, the integral $\int Q^{-1}\mathrm{d}v$ appearing in Eq. (29) can be carried out explicitly. The result is given in the following.

$$q_{\underline{a}}(a,t\,|\,a_0,t_0)=\frac{D(t-t_0)}{\sqrt{4\pi\beta\Delta(t-t_0)}\,Q^*a^b}\times\exp\left\{-\left[\frac{1-(a/a_0)^{-c}}{cQ^*a_0^c}-\mu(t-t_0)\right]^2\Big/4\beta\Delta(t-t_0)\right\}, \quad \text{for } b\neq 1$$

$$=\frac{D(t-t_0)}{\sqrt{4\pi\beta\Delta(t-t_0)}\,Q^*a^b}\times\exp\left\{-\left[\frac{\ln(a/a_0)}{Q^*}-\mu(t-t_0)\right]^2\Big/4\beta\Delta(t-t_0)\right\}, \quad \text{for } b=1 \tag{41}$$

in which $c=b-1$.

With the values of b, Q^*, μ, and β estimated above, the probability of crack exceedance, $F^*_{\underline{a}(t)}(a,t\,|\,a_0,t_0)$, can be computed using Eqs. (41) and (33) for different values of Δ. The results at $t=8000$ flight hours are plotted in Fig. 4 as the solid curve for $\Delta=8000$. It has been found[24] that this Δ value leads to the best agreement in statistical dispersion between the theoretical and experimental results for both the WPB and XWPB cases. The fractions of crack exceedance at 8000 flight hours obtained directly from the test data of 16 specimens, Fig. 2a, are also displayed in Fig. 4 as circles for comparison.

The probability distribution function of the random time to reach a given crack size a_l can be computed from Eq. (34). The result for the WPB case is plotted in Fig. 5 as solid curves for $a_l=0.254$, 0.508, and 1.016 mm (0.01, 0.02, and 0.04 in.), respectively. Also shown in Fig. 5 as circles are the corresponding test data points obtained from Fig. 2a.

It is observed from Figs. 4 and 5 that the test data correlate very well with the present statistical fatigue crack propagation model.

The same procedure outlined above was applied also to the XWPB data set. The linear regression analysis shown in Fig. 3b resulted in

$$b=0.983, \quad Q^*=2.4414\times 10^{-4}$$

$$\sigma_{\log X(t)}=0.12896, \quad \mu=1.0451, \quad \beta=0.0503 \tag{42}$$

The crack exceedance probabilities at 6000 flight hours are plotted in Fig. 6 as the solid curve based on Eq. (33) and for $\Delta=8000$. Also shown in Fig. 6 as circles are the corresponding

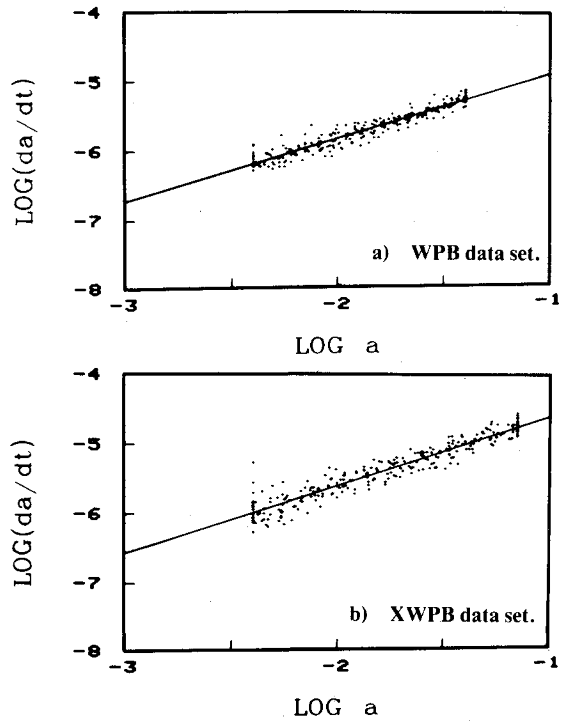

Fig. 3 Crack growth rate as a function of crack size.

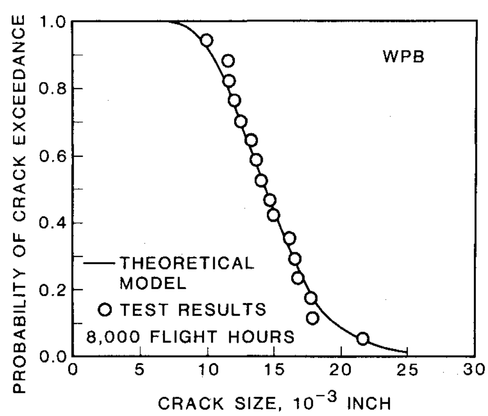

Fig. 4 Probability of crack exceedance at 8000 flight hours, WPB fastener holes.

test data points obtained from Fig. 2b. The probability distribution functions of the random times to reach crack sizes $a_l=0.203$, 0.635, and 1.778 mm (0.008, 0.025, and 0.07 in.) were computed from Eq. (34). The results are shown in Fig. 7 as solid curves. Also displayed in Fig. 7 as circles are the corresponding test results obtained from Fig. 2b. Again, Figs. 6 and 7 demonstrate excellent theoretical and experimental correlations.

Crack Propagation of Center-Cracked Specimens

Crack propagation experimental results of 64 center-cracked specimens, made of 2024-T3 aluminum, and subjected to a constant-amplitude cyclic loading, were reported in Refs. 13 and 14. The time histories of half crack lengths $\underline{a}(n)$, plotted against the number of cycles n, are shown in Fig. 8. The initial half crack length of each specimen was 9 mm and the tests were terminated when each half crack length reached 49.8 mm. The maximum cyclic load was 23.35 kN (5.2 kips) and the stress ratio was 0.2. Data for the crack growth rate $\mathrm{d}a/\mathrm{d}n$ vs the stress intensity range ΔK were obtained from the test results using the method of seven-point polynomial. The results were shown as dots in Refs. 13 and 14.

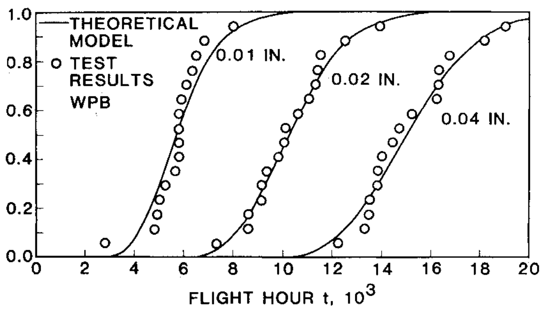

Fig. 5 Probability distribution functions of random times to reach crack sizes 0.01, 0.02, and 0.04 in.; WPB fastener holes.

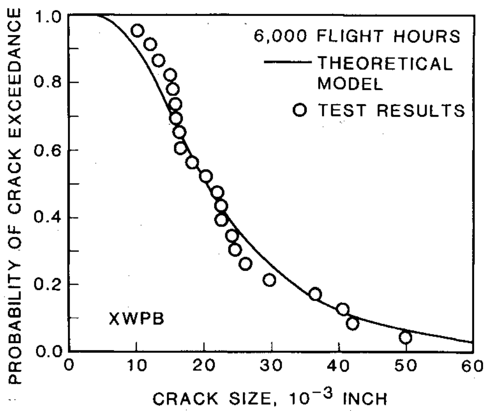

Fig. 6 Probability of crack exceedance at 6000 flight hours, XWPB data set.

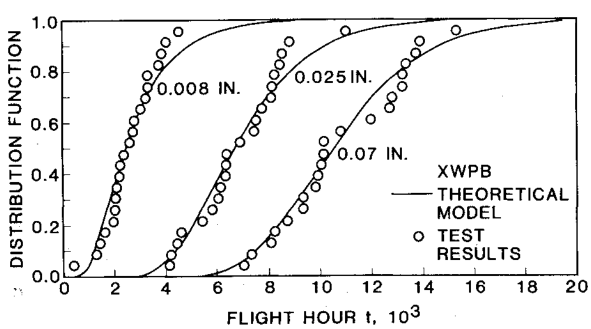

Fig. 7 Probability distribution functions of random times to reach crack sizes 0.008, 0.025, and 0.07 in.; XWPB fastener holes.

The log crack growth rate data is not linearly related to the log stress intensity range, ΔK. As a result, the following synergistic sine hyperbolic functional form is suggested for crack growth rate,[20,21]

$$\frac{\mathrm{d}\underline{a}(n)}{\mathrm{d}n} = (10)^{C_1 \sinh[C_2(\log\Delta K + C_3)] + C_4} \tag{43}$$

in which $\underline{a}(n)$ is the half crack length, ΔK the stress intensity range, and C_1, C_2, C_3, and C_4 are parameters. The randomized form for Eq. (43) is given by

$$\frac{\mathrm{d}\underline{a}(n)}{\mathrm{d}n} = X(n)\,(10)^{C_1 \sinh[C_2(\log\Delta K + C_3)] + C_4} \tag{44}$$

Taking logarithms on both sides, we obtain

$$\log\frac{\mathrm{d}\underline{a}(n)}{\mathrm{d}n} = C_1 \sinh[C_2(\log\Delta K + C_3)] + C_4 + Z(n) \tag{45}$$

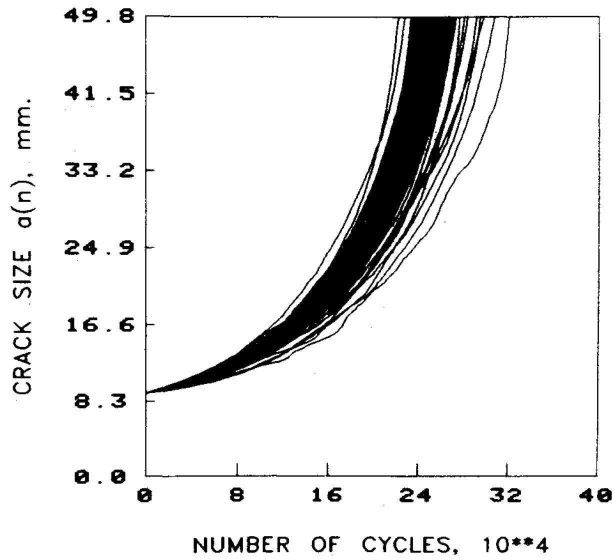

Fig. 8 Crack propagation time histories of some center-cracked specimens (after Refs. 13 and 14).

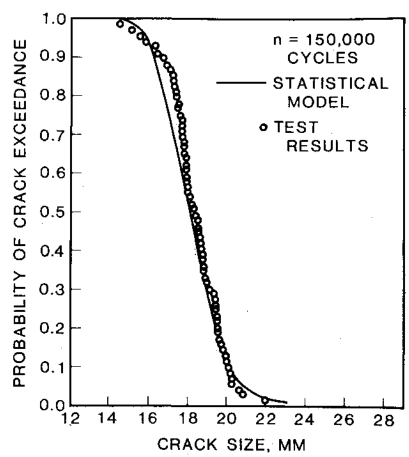

Fig. 9 Probability of crack exceedance at 150,000 cycles; center-cracked specimens.

where

$$Z(n) = \log X(n) \tag{46}$$

The stress intensity range, ΔK, for the center-cracked specimens is given by

$$\Delta K = \frac{\Delta P}{BW}\sqrt{\pi \underline{a}(n) \sec[\pi \underline{a}(n)/W]} \tag{47}$$

in which ΔP is the load range (4.16 kips), B the thickness of the specimen (0.1 in.), and W the width of the specimen (6.0 in.).

Since $X(n)$ is a stationary random process, the mean value and standard deviation are constants. Again, $Z(n) = \log X(n)$ is assumed to be normally distributed with zero mean and a standard deviation $\sigma_{Z(n)}$. From the crack growth rate data given in Refs. 13 and 14, the parameters C_1, C_2, C_3, C_4 and the standard deviation $\sigma_{Z(n)}$ can be estimated using Eq. (45) and the method of maximum likelihood. The results are

$$C_1 = 0.5, \quad C_2 = 3.4477, \quad C_3 = -1.3902$$

$$C_4 = -4.5348, \quad \sigma_{Z(n)} = 0.082345 \tag{48}$$

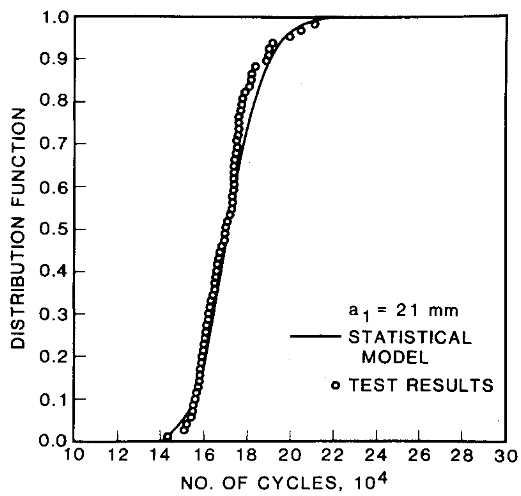

Fig. 10 Probability distribution function of random time to reach crack size 21 mm; center-cracked specimens.

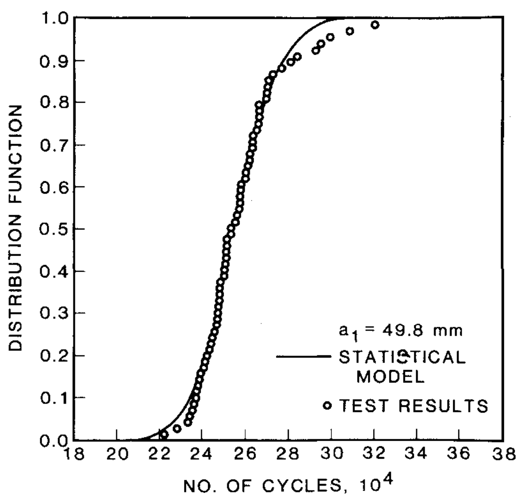

Fig. 11 Probability distribution function of random time to reach crack size 49.8 mm; center-cracked specimens.

Substitution of $\sigma_{Z(n)}$ into Eqs. (38) and (39) leads to

$$\mu = 1.0181, \qquad \beta = 0.018972 \tag{49}$$

The crack exceedance curve and the distribution of the random number of load cycles to reach a given crack length can be computed using Eqs. (29) and (33) for different values of Δ. The crack exceedance curve at 150,000 load cycles is shown in Fig. 9 as the solid curve corresponding to $\Delta = 30{,}000$. Also plotted in Fig. 9 as circles, for the purpose of comparison, are the experimental data points obtained from Fig. 8. The distribution functions for the random number of load cycles to reach half crack lengths 21 and 49.8 mm are displayed as solid curves in Fig. 10 and 11, respectively, also for $\Delta = 30{,}000$. The corresponding experimental test results are shown in these figures as circles. It is seen from Figs. 9-11 that the correlation between the theoretical and test results is also very reasonable.

Concluding Remarks

A stochastic model is proposed to analyze the uncertainty in fatigue crack propagation by introducing a random process factor to the crack propagation rate in consistence with the concept of fracture mechanics. Analytical solutions have been obtained for the probability distribution of the crack size at any given time instant and the probability distribution of the random time at which a given crack size is reached. The key to success is the approximation of the random crack size by a Markov process. While this approximation introduces an error associated with the crack propagation rate, the error is negligible if the tendency for drift dominates the tendency for diffusion, which is the case for practical applications except for extremely short fatigue lives. The application of the theory requires the estimation of only a few fracture mechanics parameters from the crack growth rate data, which is desirable in view of the limited experimental data base at the present time. It is shown in three physical examples that the theoretical results correlate very well with experimental data points in both the small crack size case and the large crack size case.

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